STEADY MOTION OF A GAS BUBBLE IN A LIQUID

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A method is proposed for calculating the velocity of a gas bubble during a steady lift in a liquid, on the assumption that the bubble transmits to the medium an array of weak waves.

In the design of industrial bubblers it is necessary to determine the length of time the gaseous and the liquid phases remain in contact. It has been established that shortly after the injection ($\sim 10^{-5}$ sec) a gas bubble begins to move at a constant velocity, i.e., the lift force is balanced by the forces of inertial and viscous drag in the liquid.

The motion of a solid sphere in a liquid was first studied theoretically by Stokes [1], who has also derived a formula for the velocity of steady motion applicable at values of the Reynolds number $Re \ll 1$. Oseen [2] has refined Stokes' solution by using for the drag coefficient two terms of the series expansion in powers of Re; his solution is valid when $Re \sim 1$. Hadamard and Rybczynski have extended this analysis to include bubbles and droplets, assuming that slip flow occurs at the interphase boundary [3]. These authors assumed that the tangential components of velocity as well as the normal and tangential stresses are, respectively, equal on both sides of the interphase boundary. Such an assumption definitely implies gas circulation inside the bubble. This process was then analyzed by Hill [4].

By introducing the concept of dynamic surface stresses, Boussinesq [5] hoped to account for the nonuniformity of stresses at the interphase boundary. For a small bubble radius R or for a high Boussinesq surface viscosity this solution becomes identical to the Stokes solution.

Levich [6] was the first one to analyze the motion of a gas bubble at a Reynolds number $Re > 1$, assuming zero tangential stresses at the interphase boundary. He has demonstrated that the velocity distribution around a bubble is not much different in viscous and in ideal liquids. Therefore, the velocity distribution in an ideal liquid was used for the calculation of the drag force and the lift velocity of a bubble. On the same premise, Moore [7] has solved this problem by a somewhat different method.

Applying the flow function to the motion of gas inside and using Levich's concept of a thin boundary layer at the bubble surface, Chao has determined the lift velocity of a bubble with gas circulation. Chao's formula is analogous to Moore's formula, but differs from the latter by including a term which depends on the gas viscosity and density inside the bubble. Various aspects of the problem were considered by Miyagi [9] as well as by Bond and Newton [10], also by other authors.

The final result of each of these researches was a relation between the constant lift velocity and the radius of a bubble. The proposed formulas agree closely enough with test data up to $Re \sim 400$. However, none of them fits the test data over any wider range of the Reynolds number. This limitation of the theories developed so far is hardly due to an incomplete consideration of viscosity effects: it is easy to see that, no matter how far the calculation of viscous drag forces were to be refined, the resulting corrections of the formula structure would be much smaller than those called for. Consequently, more appropriate solutions to the problem must be sought by a different approach.

The basic deficiency of those proposed theories is, in our opinion, that they have ignored the undular character of interaction between a moving bubble and the ambient medium. This conclusion is based on the generally valid assumption that the pressure is low when a bubble moves at a low relative velocity and that,

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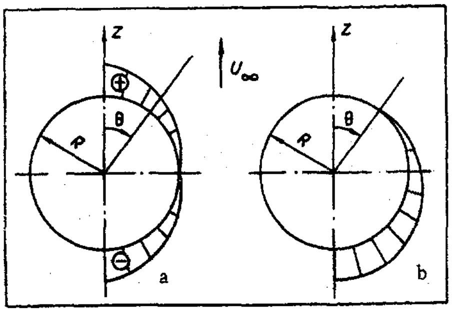

Fig. 1

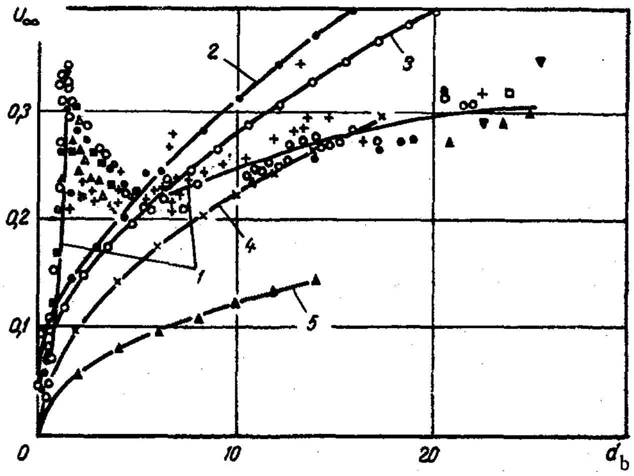

Fig. 2

Fig. 1. Distribution of (a) pressure (normal) over the bubble surface and (b) tangential velocity component in the adjoining liquid layer, at the instant when a weak pulse is applied.

Fig. 2. Lift velocity U_∞ (m/sec) of air bubbles in water: 1) approximation of test data; 2) according to formula (8) for a spherical bubble (diameter d_b, mm); for ellipsoidal bubbles according to formula (21) with: 3) $\varepsilon = 0.2$; 4) 0.5; 5) 0.8.

therefore, the liquid may be considered incompressible. It seems just as probable and is thus correct to assume that at low lift velocities there exist no eddies in the ambient medium around a bubble. These two hypotheses yield a paradoxical pressure distribution over the surface of a body (d'Alembert's paradox): the net pressure force on a bubble equal to zero. A pressure distribution according to this solution to the problem is paradoxical not only on account of a zero drag on a moving bubble but also because it implies that the pressure coefficient will be negative at the leading hemisphere. Furthermore, for any pressure in the ambient medium there will, theoretically, be some velocity at which the pressure at the leading hemisphere becomes negative. This solution was obtained on the assumption of a potential flow; the flow could be eddy, however, and not only because of friction forces but also because of normal pressure forces. For example, the appearance of eddies always accompanies the motion of a body in a medium inside a closed vessel. The basic problem then seems to be the determination of normal pressures.

We will assume that the flow around a bubble is nonseparative, i.e., an elementary jet starting at the point $\Theta = 0$, $r = R$ terminates at the point $\Theta = \pi$, $r = R$. As is well known, the pressure at the front of an acoustic wave is proportional to the density of the medium ρ, the velocity of sound in the medium c, and the increase in the velocity of the medium Δu. This velocity increment is equal to the projection of the lift velocity on the radial axis: $\Delta u = U_\infty \cos \Theta$. The leading hemisphere transmits compression waves and the trailing hemisphere transmits rarefaction waves, as during displacement. A rigorously constructed theory of motion in a liquid would have to consider that an array of waves is transmitted by the body during this motion. The solution of such a problem requires a special study, however, because of its complexity. Formulas for the drag force on a bubble and for the velocity of steady lift will be derived here on the basis of a qualitatively novel but also simpler assumption.

We consider a spherical gas bubble (Fig. 1) which is initially at standstill in a liquid and then receives a short weak impulse. The upper hemisphere tends to compress the adjoining medium, whereas the lower hemisphere tries to separate from the adjoining liquid. A compression wave will travel upward and a rarefaction wave will travel downward. It is assumed that the liquid remains continuous everywhere. A velocity jump will occur at the wave fronts. If the velocity of the bubble center is U_∞, then the projection of the absolute velocity of a liquid particle on the radial axis at a given point will vary from 0 to Δu. The pressure distribution p over the bubble surface must be analogous. The maximum absolute value of pressure p_{max}, at the uppermost point and at the lowermost point of the sphere, is unknown. The net normal pressure force is

$$N = 2\int_0^{\pi/2} p_{\max}\cos^2\Theta R^2 \sin\Theta d\Theta = \frac{4}{3}\pi R^2 p_{\max}. \quad (1)$$

We will now explore the structure of p_{max}. In order to be able to move, a bubble must perform work on displacing a liquid particle from the uppermost point to the lowermost point of the surface. Moreover, this must transpire within a time $t_1 = 2R/U_\infty$, i.e., while the bubble is lifted through a distance equal to its diameter. The acceleration of a particle moving down along the bubble surface is

$$a = \frac{-1}{\rho R}\frac{dp}{d\Theta} = \frac{p_{max}\sin\Theta}{\rho R}. \tag{2}$$

The density of the liquid ρ may be considered constant here. The velocity of such a particle is

$$u = u_0 + \int_0^t a(\tau)\,d\tau = u_0 + \int_0^\Theta a(\vartheta)\,d\vartheta, \tag{3}$$

with u_0 denoting the initial velocity (at the uppermost point) which is to be equated to zero. Differentiating (3) with respect to the upper limit of the integral, then inserting (2) into the obtained equation, and integrating the result will finally yield

$$u = 2\sin\frac{\Theta}{2}\sqrt{\frac{p_{max}}{\rho}}. \tag{4}$$

The travel time of a particle down along the bubble surface is

$$t = \frac{1}{2\sqrt{\frac{p_{max}}{\rho}}}\int_0^\pi \frac{d\vartheta}{\sin\frac{\vartheta}{2}} = t_1 = \frac{2R}{U_\infty}. \tag{5}$$

It follows from (5) that the pressure at the uppermost point of the bubble is proportional to the velocity head. The numerical value of p_{max} cannot be found because, when the limits are inserted, the integral in the equation tends to infinity: an infinitely long time of particle travel from the uppermost point to the lowermost point on the bubble surface, the initial velocity of such a particle being zero. This situation is encountered also in classical analyses of the problem. Furthermore, by virtue of the continuity of the velocity distribution in the medium, the motion of particles along the nearest streamlines is very slow. This is very interesting: if the liquid is considered ideal, then masses of it must adjoin the bubble both at the top and at the bottom. The problem of determining these masses and analyzing their behavior becomes of fundamental importance here.

In our solution we let the maximum pressure be equal to the velocity head:

$$p_{max} = \frac{\rho U_\infty^2}{2}. \tag{6}$$

We then equate the total drag force and the lift force

$$\frac{4}{3}\pi R^2\frac{\rho U_\infty^2}{2} = \frac{4}{3}\pi R^3\rho g. \tag{7}$$

From this we obtain the steady lift velocity of a bubble (g denotes the acceleration of free fall):

$$U_\infty = \sqrt{2gR} \cong 4.43\sqrt{R}, \text{ m/sec}. \tag{8}$$

Values of U_∞ according to (8) are compared in Fig. 2 with test data from [11]. The agreement is generally more satisfactory than with earlier formulas. We will explain some discrepancies between theory and experiment.

Bubbles 2 mm in diameter or larger cease to be spherical. The flatness of the $U_\infty = f(R)$ test curves confirms the validity of the preceding analysis: if a spherical bubble has become ellipsoidal (with the same volume), then friction does not change as much as the drag due to normal pressure force increases.

We will construct an analogous scheme for calculating the steady lift velocity of a bubble shaped as an ellipsoid of revolution. The equation of an ellipse in polar coordinates with the origin at one focus is

$$r = \frac{\gamma}{1-\varepsilon\cos\varphi}, \tag{9}$$

where r denotes the radius-vector of a point on the ellipse, $\varphi = [(\pi/2)-\Theta]$ (angle Θ is indicated in Fig. 1); $\varepsilon = c/a$ denotes the eccentricity; a and b are the major (horizontal) and the minor (vertical) semiaxis, respectively; $c = \sqrt{a^2-b^2}$; and $\gamma = b^2/a$ is another parameter. Angle φ from a point on the equator to the uppermost point of the bubble varies from 0 to

$$\varphi_0 = \operatorname{arctg}\frac{b}{c} = \operatorname{arctg}\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}. \tag{10}$$

The angle between the tangent to the curve and the polar axis is

$$\alpha = \operatorname{arctg}\frac{\dfrac{dr}{d\varphi} + r\operatorname{ctg}\varphi}{\operatorname{ctg}\varphi\dfrac{dr}{d\varphi} - r}, \tag{11}$$

where

$$\frac{dr}{d\varphi} = \frac{-b^2\varepsilon\sin\varphi}{a(1-\varepsilon\cos\varphi)^2}. \tag{12}$$

We insert (12) into (11) and use the result for calculating the projection of the normal pressure force on the vertical axis:

$$p_b = p\cos\alpha = \frac{p[r(\varphi),\ \varphi]}{\sqrt{1+\left(\dfrac{\varepsilon-\cos\varphi}{\sin\varphi}\right)^2}} = \frac{p(\varphi)\sin\varphi}{\sqrt{1-2\varepsilon\cos\varphi+\varepsilon^2}}. \tag{13}$$

The projection of the velocity on the normal to the ellipse at any point is

$$U_n = U_\infty\cos\alpha. \tag{14}$$

Therefore, we have for p_b

$$p_b = \frac{p_{\max}\sin^2\varphi}{1-2\varepsilon\cos\varphi+\varepsilon^2}. \tag{15}$$

The deformation of a bubble does not change its volume:

$$\frac{4}{3}\pi a^2 b = \frac{4}{3}\pi R^3. \tag{16}$$

Using the last expression and the expression for the eccentricity, we obtain

$$a = \frac{R}{\sqrt[6]{1-\varepsilon^2}};\quad b = R\sqrt[3]{1-\varepsilon^2}. \tag{17}$$

The net force of normal pressure is found according to the expression

$$N = 2\int_0^{\varphi_0} 2\pi\left(r\cos\varphi - \sqrt{a^2-b^2}\right)\frac{rp_b(\varphi)\,d\varphi}{\sin(\varphi-\alpha)}$$

$$= p_{\max}4\pi R^2\sqrt[3]{(1-\varepsilon^2)^2}\int_0^{\varphi_0}\frac{(\cos\varphi-\varepsilon)\sin^2\varphi\,d\varphi}{(1-\varepsilon\cos\varphi)^4\sqrt{1-2\varepsilon\cos\varphi+\varepsilon^2}}. \tag{18}$$

As before, we now find the time in which a liquid particle moves from the uppermost point to the lowermost point under a pressure distribution $p = p_{\max}\cos\varphi$, and we then equate it to the time in which the bubble has been lifted through a distance equal to its minor (vertical) axis. From this equation follows the structure of $p_{\max}$:

$$p_{\max} = \frac{1-\varepsilon^2}{4}\,\frac{\rho U_\infty^2}{2}\left[\int_{-\varphi_0}^{\varphi_0}\frac{(1-2\varepsilon\cos\varphi+\varepsilon^2)^{3/4}d\varphi}{(1-\varepsilon\cos\varphi)^2(\sqrt{1-2\varepsilon\cos\varphi+\varepsilon^2}-\sin\varphi)^{1/2}}\right]^2. \tag{19}$$

According to (19), p_{max} is proportional to the velocity head and depends on the eccentricity of the elliptical bubble section. Letting $p_{max} = \rho U_\infty^2/2$ and equating the force N to the lift force, we have

$$\frac{\rho U_\infty^2}{2} 4\pi R^2 \sqrt[3]{(1-\varepsilon^2)^2} \int_0^{\varphi_0} \frac{(\cos\varphi - \varepsilon)\sin^2\varphi d\varphi}{(1-\varepsilon\cos\varphi)^4 \sqrt{1-2\varepsilon\cos\varphi+\varepsilon^2}} = \frac{4}{3}\pi R^3 \rho g. \tag{20}$$

This yields the steady lift velocity of a bubble:

$$U_\infty = \frac{\sqrt{\frac{2}{3} g}\sqrt{R}}{\sqrt[3]{1-\varepsilon^2}\sqrt{\int_0^{\varphi_0} \frac{(\cos\varphi - \varepsilon)\sin^2\varphi d\varphi}{(1-\varepsilon\cos\varphi)^4 \sqrt{1-2\varepsilon\cos\varphi+\varepsilon^2}}}}. \tag{21}$$

The velocity U_∞ calculated according to formula (21), as a function of the initial radius R and the eccentricity ε of a bubble in motion, is shown in Fig. 2. The effect of ε is quite appreciable. For a bubble with a radius R = 5 mm, for instance, U_∞ = 30.1 cm/sec when ε = 0.1, a = 5.01 mm, and b = 4.98 mm, but U_∞ = 11.9 cm/sec when ε = 0.8, a = 5.93 mm, and b = 4.55 mm: the lift velocity should thus decrease 2.5 times.

In conclusion, we note that the true shape of a gas bubble moving in a liquid is not ellipsoidal but more intricate. The data presented here can be useful in problems concerning the motion of gas bubbles as well as of other bodies in a liquid.

NOTATION

Re	is the Reynolds number;
Θ	is an angle, deg;
r	is the instantaneous radius, m;
R	is the radius of a spherical bubble, m;
ρ	is the density of the liquid, kg/m^3;
c	is the velocity of sound in the liquid, m/sec;
Δu	is the increment of velocity of the liquid acted on by the bubble surface, m/sec;
U_∞	is the steady lift velocity of a bubble, m/sec;
p	is the pressure, N/m^2;
N	is the net force of normal pressure, N;
t	is the time, sec;
a	is the acceleration, m^2/sec;
u	is the velocity of a liquid particle, m/sec;
g	is the acceleration of free fall, m/sec^2;
d_b	is the bubble diameter, mm;
p_{max}	is the maximum pressure on the bubble surface, N/m^2;
τ	is the time (in the integration), sec;
ϑ	is an angle (in the integration), deg;
ε	is the eccentricity;
γ	is a parameter, m;
a, b	are the semiaxes of an ellipse;
$c = a^2 - b^2$;	
φ	is the polar angle, deg;
α	is the slope angle of a tangent to the ellipse, deg.

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